Improving the Fidelity of Quantum Cloning by Fast Cycling away the Unwanted Transition

Shubhrangshu Dasgupta and G. S. Agarwal
Physical Research Laboratory, Navrangpura, Ahmedabad - 380 009, India
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The fidelity of quantum cloning is very often limited by the accompanying unwanted transitions. We show how the fidelity can be improved by using a coherent field to cycle away the unwanted transitions. We demonstrate this explicitly in the context of the model of Simon *et al.* [J. Mod. Opt. **47**, 233 (2000) ; Phys. Rev. Lett. **84**, 2993 (2000)]. We also investigate the effects of the number of atoms on the quality of quantum cloning. We show that the universality of the scheme can be maintained by choosing the cycling field according to the input state of the qubit.

I. INTRODUCTION

Quantum no-cloning theorem states that it is impossible to clone perfectly an arbitrary *unknown* pure quantum state [1], or a mixed quantum state [2]. The origin of this theorem can be traced to the linearity of quantum mechanics. However imperfect quantum cloning is possible. An optimal $1 \rightarrow 2$ imperfect cloner has been proposed [3], which is universal for all input qubits and is compatible with no-signalling constraints too [4]. It has been generalized [5,6] for the case of $N \rightarrow M$ $(M \geq N)$ cloner. The upper bound for the fidelity of a $N \rightarrow M$ cloner has been established [7]. Results are also available for the optimal cloning of arbitrary pure and mixed states in d-dimensions $(d \geq 2)$ [6,8]. Besides, cloning of entangled states [9] and of Gaussian-distributed quantum variables [10] has been considered. Quantum cloning was originally discussed in terms of polarization states of photons [1,11,12]. Making use of two two-level atoms with orthogonal transition dipole moments, Mandel [12] proved that one can make the output of the photon amplifier independent of the input polarization. Perfect cloning of photonic states has been proved impossible due to inevitable coexistence of spontaneous emission with stimulated emission process [11,12]. Simon and co-workers [13,14] have proposed a quantum cloning machine (QCM) consisting of three-level atoms. They have shown that the quantum cloning of a single input qubit (polarization state of a photon) using a V-system is possible with a fidelity 5/6 at least for shorter interaction times. This value is optimal for a $1 \rightarrow 2$ quantum cloner [3]. Similar results for Λ-systems have been reported [15]. Finally note that the state-dependent cloning has also been studied extensively [16]. In this paper, we address the question if the fidelity of a V-system cloner can be improved by using some type of external field.

The organization of this paper is as follows. In Sec. II, we briefly outline the cloning scheme introduced by Simon *et al.*. We prove the universality of their scheme by using a new basis for the states of the radiation fields. In Sec. III, we examine the reason for imperfect fidelity in a V-scheme and introduce a way to improve this fidelity by using a coherent field. The external field cycles away the unwanted transition responsible for spontaneous emission. We present both analytical and numerical results for the fidelity. We discuss the question of the universality of the scheme. In Sec. IV, we examine the question of improvement in fidelity by considering a cloner consisting of two V-systems. In Sec. V, we demonstrate the improvement in average fidelity for the case of a fixed cycling field for all input states of the qubit.

II. QUANTUM CLONING BASED ON STIMULATED EMISSION IN A V-SYSTEM

A. Optimal Photon Cloner with a V-configuration

Recently Simon *et al.* [13,14] have proposed a new scheme for quantum cloning of a photonic qubit. They considered a cloning device consisting of an ensemble of atoms trapped inside a cavity. The relevant atomic transitions correspond to the V-system. These are three-level systems with two degenerate excited states $|e_1\rangle$ and $|e_2\rangle$ and a common ground level $|g\rangle$. The ground level is coupled to the excited states by two orthogonal field modes a_1 and a_2, respectively.

In the interaction picture [17], the effective Hamiltonian under dipole and rotating wave approximations [18,19] can be written as

$$H_I = \hbar g \sum_{k=1}^{N} (\sigma_{+1}^k a_1 + \sigma_{+2}^k a_2) + \text{ H. c.}, \tag{1}$$

where g is the coupling constant between the field-modes and the atoms, $\sigma_{+1(2)}^k$'s $[= (|e_{1(2)}\rangle\langle g|)_k]$ are the raising operators between the corresponding states of the k-th atom. Here g is assumed to be equal for all the atoms [20]. Also both the cavity-modes are assumed to be resonant with the corresponding atomic transitions.

Let each atom be prepared initially in a mixed excited state

$$\rho = \frac{1}{2}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|) \tag{2}$$

and the photonic qubit be prepared in a state $a_1^\dagger|0,0\rangle \equiv |1,0\rangle$. The time-development operator $U = e^{-iH_I t}$ will provide the time-evolution of the entire (atom+photon) system and this is used to study quantum cloning.

The quality of cloning is characterized in terms of fidelity [21], which can be defined [13] in the following way:

$$F = \sum_{k=0}^{N+1} \sum_{l=0}^{N} p(k,l) \left(\frac{k}{k+l} \right). \tag{3}$$

Here $p(k,l)$ represents the probability of finding k photons in the initial mode and l photons in the orthogonal mode a_2 in the evolved state. It should be noted that for an ensemble of N atoms, the maximum value of k will be $N+1$, which corresponds to all the atoms decaying to the ground state through the emission of the a_1-photon. Thus F is a kind of an average of the relative frequency of photons in initial mode a_1 in the final state.

As shown by Simon *et al.*, the fidelity is optimal for short interaction times and for $N = 6$. It decreases for later times. They have explained this behavior in terms of stimulated and spontaneous emissions on the transitions $|e_1\rangle \to |g\rangle$ and $|e_2\rangle \to |g\rangle$, respectively. If there is an extra photon in a_1-mode, it can be considered as a clone of the initial qubit. Note that the probability to get a clone is reduced if there is an extra photon in the other (a_2) mode, which is due to spontaneous emission.

B. Question of Universality of Cloning by a V-system

It is mentioned in Ref. [14] that the above scheme is universal, i.e., the V-system cloner can clone even any arbitrary photonic qubit, say, $(\alpha a_1^\dagger + \beta a_2^\dagger)|0,0\rangle$ with the same non-unity fidelity. Simon *et al.* argued that this is because the initial mixed state and the Hamiltonian are invariant under a unitary transformation. In what follows, we demonstrate explicitly this universality by changing the basis to a general qubit state.

Consider a single atom in V-configuration, initially prepared in an incoherent superposition of the two excited states. We will work with wavefuntions and hence we use an initial state

$$|s\rangle = \frac{1}{\sqrt{2}} \left(|e_1\rangle + e^{i\theta}|e_2\rangle \right). \tag{4}$$

and average the final results over all possible values of the parameter θ. Let the photon be in a superposition state

$$b_1^\dagger|0,0\rangle \equiv \left(\alpha a_1^\dagger + \beta a_2^\dagger \right) |0,0\rangle \equiv \alpha|1,0\rangle + \beta|0,1\rangle \tag{5}$$

as well, where α and β are the complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. Note that an average over θ will give an initial state of the atom, which is a mixed state. We consider the photon as a qubit [22], which can be in any linear superposition of the two orthogonal states. Let us define the basis state $b_2^\dagger|0,0\rangle$, which is orthogonal to (5). The new operators b_1 and b_2 must satisfy the commutation relations

$$[b_1, b_2] = [b_1, b_2^\dagger] = 0. \tag{6}$$

Using Eqs. (5) and (6), we get

$$b_2^\dagger \equiv -\beta^* a_1^\dagger + \alpha^* a_2^\dagger. \tag{7}$$

The time-evolution of the entire system is determined by the Schrödinger equation

$$i\hbar \frac{d}{dt}|\Psi(t)\rangle = H_I|\Psi(t)\rangle, \tag{8}$$

where H_I is given by Eq. (1) for $N = 1$. We expand $|\Psi(t)\rangle$ in terms of the relevant basis states. Starting with the initial conditions [Eqs. (4) and (5)], these relevant states were found to be

$$|e_1\rangle|1,0\rangle \ ; \ |e_2\rangle|1,0\rangle \ ; \ |g\rangle|2,0\rangle \ ; \ |g\rangle|1,1\rangle \ ; \ |g\rangle|0,2\rangle \ ; \ |e_1\rangle|0,1\rangle \ ; \ |e_2\rangle|0,1\rangle. \tag{9}$$

The only non-zero expansion amplitudes C_α^{mn} at time $t = 0$ are

$$C_{e_1}^{10}(0) = \frac{\alpha}{\sqrt{2}} \ ; \ C_{e_1}^{01}(0) = \frac{\beta}{\sqrt{2}} \ ; \ C_{e_2}^{10}(0) = \frac{\alpha}{\sqrt{2}}e^{i\theta} \ ; \ C_{e_2}^{01}(0) = \frac{\beta}{\sqrt{2}}e^{i\theta}, \tag{10}$$

where the subscript (superscript) denotes the atom (photons) in the state α (m, n). Then all the expansion amplitudes can be evaluated in closed form with the following results:

$$\begin{aligned}
C_{e_1}^{10}(t) &= \frac{\alpha}{\sqrt{2}} \cos\left(\sqrt{2}gt\right), \\
C_g^{20}(t) &= -i\frac{\alpha}{\sqrt{2}} \sin\left(\sqrt{2}gt\right), \\
C_{e_2}^{10}(t) &= \frac{1}{2\sqrt{2}} \left[\left(\alpha e^{i\theta} - \beta\right) + \left(\beta + \alpha e^{i\theta}\right) \cos\left(\sqrt{2}gt\right)\right], \\
C_{e_1}^{01}(t) &= \frac{1}{2\sqrt{2}} \left[\left(\beta - \alpha e^{i\theta}\right) + \left(\beta + \alpha e^{i\theta}\right) \cos\left(\sqrt{2}gt\right)\right], \\
C_g^{11}(t) &= -\frac{i}{2} \left(\beta + \alpha e^{i\theta}\right) \sin\left(\sqrt{2}gt\right), \\
C_g^{02}(t) &= -i\frac{\beta}{\sqrt{2}}e^{i\theta} \sin\left(\sqrt{2}gt\right), \\
C_{e_2}^{01}(t) &= \frac{\beta}{\sqrt{2}}e^{i\theta} \cos\left(\sqrt{2}gt\right).
\end{aligned} \tag{11}$$

The reduced density matrix of the field is defined by

$$\rho_F = \mathrm{Tr}_A(|\Psi(t)\rangle\langle\Psi(t)|). \tag{12}$$

Using Eq. (12), the probability $\tilde{p}(k, l)$ that k photons will be in b_1-mode and l photons in b_2-mode can be written in terms of b-operators as

$$\tilde{p}(k, l) = \langle 0, 0| \frac{b_1^k \, b_2^l}{\sqrt{k! \, l!}} \, \rho_F \, \frac{b_1^{\dagger k} \, b_2^{\dagger l}}{\sqrt{k! \, l!}} |0, 0\rangle. \tag{13}$$

Further in order to get the initial atomic state used by Simon *et al.*, we average $\tilde{p}(k, l)$ over all values of θ. A lengthy derivation yields the following:

$$\tilde{p}_a(2, 0) = \frac{1}{2} \sin^2\left(\sqrt{2}gt\right), \tag{14a}$$

$$\tilde{p}_a(1, 1) = \frac{1}{4} \sin^2\left(\sqrt{2}gt\right), \tag{14b}$$

$$\tilde{p}_a(0, 1) = \frac{1}{8} \cos^2\left(\sqrt{2}gt\right) - \frac{1}{4} \cos\left(\sqrt{2}gt\right) + \frac{1}{8}, \tag{14c}$$

$$\tilde{p}_a(1, 0) = \frac{5}{8} \cos^2\left(\sqrt{2}gt\right) + \frac{1}{4} \cos\left(\sqrt{2}gt\right) + \frac{1}{8}, \tag{14d}$$

where $\tilde{p}_a(k, l)$ is the θ-averaged value of $\tilde{p}(k, l)$. The Eqs. (14) lead to the following expression for the fidelity:

$$F = \tilde{p}_a(1, 0) + \tilde{p}_a(2, 0) + \frac{1}{2}\tilde{p}_a(1, 1) = \frac{3}{4} + \frac{1}{4} \cos(\sqrt{2}gt). \tag{15}$$

Clearly the fidelity does not depend upon α and β. This reflects the fact that the V-scheme is universal as a cloner, which can clone even a general superposition of two orthogonal modes of the field, albeit imperfectly. A similar result has been reported recently using a different method [15].

III. A METHOD TO IMPROVE THE FIDELITY OF THE V-SCHEME FOR ARBITRARY STATE OF THE INPUT PHOTON

It is clear that the fidelity of cloning is degraded by the emission of a photon of the "wrong" type. Thus to improve the fidelity one should reduce the probability $p(1,1)$ and $p(0,1)$ of emitting in the mode a_2, which is caused by the atomic population in $|e_2\rangle$-level. One possible way of doing this is to *cycle* this population away, so that this unwanted spontaneous decay does not occur very often. We show that it can be done by applying a classical pump field, which causes the population in the state $|e_2\rangle$ to pulsate between a metastable state $|f\rangle$ and $|e_2\rangle$. However any biasing by the external field is likely to take away the system from universality. We *get over the problem* by making the bias dependent on the state to be cloned. The scheme would then become *near universal*.

Consider a four-level atomic configuration as shown in Fig. 1. The excited states $|e_1\rangle$ and $|e_2\rangle$ are coupled to the common ground level $|g\rangle$ through the two orthogonal modes a_1 and a_2 of the quantized electromagnetic field, respectively. The coupling constant between each of these excited states and $|g\rangle$ is g. We consider the action of classical fields coupling the state $|e_i\rangle$ $(i = 1, 2)$ with the metastable state $|f\rangle$. The corresponding Rabi frequency is $2gG_i$, where G_i is a multiplying factor and is related to the number of photons in the classical field. We assume all the fields to be resonant with the corresponding atomic transitions.

We start with a single atom prepared in the state $|s\rangle$ [Eq. (4)]. The initial photonic qubit is in b_1-mode. We will work in the interaction picture. Then using the rotating wave approximation to eliminate the fast-oscillating energy non-conserving terms, we obtain the effective Hamiltonian as

$$H_I = \hbar g \left[\sigma_{+1} a_1 + \sigma_{+2} a_2 + \sum_{i=1,2} G_i |e_i\rangle\langle f| \right] + \text{H.c.}, \tag{16}$$

where $\sigma_{+1(2)} = |e_{1(2)}\rangle\langle g|$ are the raising operators of the atom as defined in the previous section.

In order to understand how the scheme can be made *near universal*, we rewrite (16) in terms of the b_1 and b_2-modes and redefined atomic states $|e_1'\rangle$ and $|e_2'\rangle$ as

$$H_I = \hbar g \left[|e_1'\rangle\langle g| b_1 + |e_2'\rangle\langle g| b_2 + G_1' |e_1'\rangle\langle f| + G_2' |e_2'\rangle\langle f| \right] + \text{H.c.}, \tag{17}$$

where

$$|e_1'\rangle = \alpha|e_1\rangle + \beta|e_2\rangle \ ; \ |e_2'\rangle = \alpha^*|e_2\rangle - \beta^*|e_1\rangle \ ; \ G_1' = \alpha^* G_1 + \beta^* G_2 \ ; \ G_2' = -\beta G_1 + \alpha G_2. \tag{18}$$

Clearly if we choose

$$G_1/G_2 = -(\beta^*/\alpha^*), \tag{19}$$

then the Hamiltonian in the new basis is like a single bias field acting on the atomic transition. This analysis implies that we can deal with arbitrary input states of the qubit by just choosing the bias field appropriately [Eq. (19)]. Note also the important property

$$|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| = |e_1'\rangle\langle e_1'| + |e_2'\rangle\langle e_2'|, \tag{20}$$

so that the initial incoherent superposition remains an incoherent superposition in the primed basis. We next calculate the effect of the bias field on the fidelity of the system by setting $G_1' = 0$ in Eq. (17). Let the eigenstates of $b_1^\dagger b_1$ and $b_2^\dagger b_2$ be denoted by $|\widetilde{n,m}\rangle$. From (17) we find that the following states participate in time evolution:

$$|e_1'\rangle|\widetilde{1,0}\rangle \ ; \ |e_2'\rangle|\widetilde{1,0}\rangle \ ; \ |f\rangle|\widetilde{1,0}\rangle \ ; \ |g\rangle|\widetilde{1,1}\rangle \ ; \ |g\rangle|\widetilde{2,0}\rangle \ ; \ |e_1'\rangle|\widetilde{0,1}\rangle. \tag{21}$$

Note that the state $|e_1'\rangle|\widetilde{0,1}\rangle$ is produced by the two-step process $|e_2'\rangle|\widetilde{1,0}\rangle \to |g\rangle|\widetilde{1,1}\rangle \to |e_1'\rangle|\widetilde{0,1}\rangle$. We expand $|\Psi(t)\rangle$ in terms of these basis states and we obtain the following first order differential equations for the expansion amplitudes $\tilde{C}$'s:

$$\begin{aligned}
\dot{\tilde{C}}{}^{10}_{e_1}(t) &= -\sqrt{2} i g \tilde{C}^{20}_g(t), \\
\dot{\tilde{C}}{}^{20}_{g}(t) &= -\sqrt{2} i g \tilde{C}^{10}_{e_1}(t), \\
\dot{\tilde{C}}{}^{10}_{e_2}(t) &= -i g \tilde{C}^{11}_g(t) - i g G_2' \tilde{C}^{10}_f(t),
\end{aligned}$$

$$
\begin{aligned}
\dot{\tilde{C}}_f^{10}(t) &= -igG_2'\tilde{C}_{e_2}^{10}(t), \\
\dot{\tilde{C}}_g^{11}(t) &= -ig\tilde{C}_{e_1}^{01}(t) - ig\tilde{C}_{e_2}^{10}(t), \\
\dot{\tilde{C}}_{e_1}^{01}(t) &= -ig\tilde{C}_g^{11}(t).
\end{aligned} \tag{22}
$$

Solving those equations subject to the initial conditions

$$
\tilde{C}_{e_1}^{10}(0) = \frac{1}{\sqrt{2}} \quad ; \quad \tilde{C}_{e_2}^{10}(0) = \frac{1}{\sqrt{2}}e^{i\theta}, \tag{23}
$$

results

$$
\begin{aligned}
\tilde{C}_g^{11}(t) &= A\sin\left(\frac{\Omega_1}{\sqrt{2}}gt\right) + B\sin\left(\frac{\Omega_2}{\sqrt{2}}gt\right), \\
\tilde{C}_f^{10}(t) &= \frac{1}{2G_2'}\left[(\Omega_1^2-4)A\sin\left(\frac{\Omega_1}{\sqrt{2}}gt\right) + (\Omega_2^2-4)B\sin\left(\frac{\Omega_2}{\sqrt{2}}gt\right)\right], \\
\tilde{C}_{e_2}^{10}(t) &= \frac{i}{2\sqrt{2}G_2'^2}\left[\Omega_1(\Omega_1^2-4)A\cos\left(\frac{\Omega_1}{\sqrt{2}}gt\right) + \Omega_2(\Omega_2^2-4)B\cos\left(\frac{\Omega_2}{\sqrt{2}}gt\right)\right], \\
\tilde{C}_{e_1}^{01}(t) &= -\frac{i}{2\sqrt{2}G_2'^2}\left[\Omega_1(\Omega_1^2-2G_2'^2-4)A\cos\left(\frac{\Omega_1}{\sqrt{2}}gt\right) + \Omega_2(\Omega_2^2-2G_2'^2-4)B\cos\left(\frac{\Omega_2}{\sqrt{2}}gt\right)\right], \\
\tilde{C}_{e_1}^{10}(t) &= \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}gt\right), \\
\tilde{C}_g^{20}(t) &= -\frac{i}{\sqrt{2}}\sin\left(\sqrt{2}gt\right),
\end{aligned} \tag{24}
$$

where

$$
\Omega_1 = \left(G_2'^2 + 2 + \sqrt{G_2'^4+4}\right)^{\frac{1}{2}} \quad , \quad \Omega_2 = \left(G_2'^2 + 2 - \sqrt{G_2'^4+4}\right)^{\frac{1}{2}},
$$

$$
\begin{aligned}
A &= \frac{1}{2}ie^{i\theta}\frac{(\Omega_2^2-2G_2'^2-4)}{\Omega_1\sqrt{G_2'^4+4}}, \\
B &= -\frac{1}{2}ie^{i\theta}\frac{(\Omega_1^2-2G_2'^2-4)}{\Omega_2\sqrt{G_2'^4+4}}.
\end{aligned} \tag{25}
$$

Using the relation (12), we get the reduced density matrix of the field. The diagonal element of this density matrix in field basis $|\widetilde{k,l}\rangle$ gives the probability $\tilde{p}(k,l)$ that k photons are in b_1-mode and l photons are in b_2-mode and these are given by

$$
\begin{aligned}
\tilde{p}(2,0) &= |\tilde{C}_g^{20}(t)|^2 = \frac{1}{2}\sin^2\left(\sqrt{2}gt\right), \\
\tilde{p}(1,1) &= |\tilde{C}_g^{11}(t)|^2, \\
\tilde{p}(1,0) &= |\tilde{C}_f^{10}(t)|^2 + |\tilde{C}_{e_2}^{10}(t)|^2 + |\tilde{C}_{e_1}^{10}(t)|^2, \\
\tilde{p}(0,1) &= |\tilde{C}_{e_1}^{01}(t)|^2.
\end{aligned} \tag{26}
$$

Hence the fidelity F takes the following form:

$$
F = 1 - \left[\tilde{p}_a(0,1) + \frac{1}{2}\tilde{p}_a(1,1)\right]. \tag{27}
$$

We have plotted this as a function of time for $G_2' = 0, 3$ in Fig. 2. It is found that as G_2' increases, the fidelity becomes unity more often. Whenever both $\tilde{p}_a(1,1)$ and $\tilde{p}_a(0,1)$ become zero, $F(t)$ becomes unity. In fact, by introducing the classical pump field, we cycle the atomic population in the state $|e_2'\rangle$ to the state $|f\rangle$ and back. This inhibits the spontaneous decay of the atom in the state $|e_2'\rangle$ to the ground state $|g\rangle$ irrespective of whether there is any photon

or not in "right" mode. There are two time-scales of oscillation of $F(t)$. The faster small-amplitude oscillation is attributed to that of $[\tilde{p}_a(0,1) + (1/2)\tilde{p}_a(1,1)]$. This oscillation can be increased by G_2' so that the atom effectively goes to the state $|f\rangle$ very frequently. This means that $F(t)$ becomes close to unity more frequently. The effect of spontaneous emission from the state $|f\rangle$ is ignored assuming that the time scale for this decay is much larger than that for Rabi oscillation between $|e_2'\rangle$ and $|f\rangle$-levels. In Fig. 3 we display the mean number of "right" and of all photons. This figure also shows how the improvement in cloning is obtained by the use of a cycling or bias field.

IV. FIDELITY OF CLONING WITH TWO ATOMS

In this section, we consider a cloning machine consisting of two atoms and we demonstrate improvement in the fidelity for a larger domain of times if we adopt the use of the cycling or bias field. We consider the case of two V-atoms. The interaction Hamiltonian is obtained by summing (17) and a similar Hamiltonian involving the interaction of the another atom B with the fields. The initial state of the atomic system is

$$\rho = \prod_{\mu=\mathrm{A,B}} \left(|e_1\rangle_\mu \, {}_\mu\langle e_1| + |e_2\rangle_\mu \, {}_\mu\langle e_2|\right) , \tag{28}$$

and we assume single photon in the mode b_1. We assume that we work under the condition (19) so that the Hamiltonian reduces to

$$H_I = \hbar g \sum_{\mu=\mathrm{A,B}} \left(|e_1'\rangle_\mu \, {}_\mu\langle g|b_1 + |e_2'\rangle_\mu \, {}_\mu\langle g|b_2 + G_2'|e_2'\rangle_\mu \, {}_\mu\langle f|\right) + \mathrm{H.c.}. \tag{29}$$

In order to use the wavefunction picture we use the initial condition for the atom as

$$\rho = \prod_{\mu=\mathrm{A,B}} |\Psi_\mu\rangle\langle\Psi_\mu| \qquad ; \qquad |\Psi_\mu\rangle = \frac{1}{\sqrt{2}} \left(|e_1'\rangle_\mu + e^{i\theta_\mu}|e_2'\rangle_\mu\right) , \tag{30}$$

and average the final results over θ_A and θ_B. For the two-atom case with bias field we have to use a large number of relevant basis states – the size of the basis states increase very rapidly with increase in the number of atoms. The equations for the amplitudes are solved numerically using fifth-order Runge-Kutta method. From the numerical solutions we calculate the fidelity F. It is clear from the Fig. 4 that the cycling field makes the fidelity much higher for a very large range of times. Note that the fidelity of the two-atom cloner can be expressed as

$$F(t) = 1 - \left[\frac{1}{3}\tilde{p}_a(2,1) + \frac{2}{3}\tilde{p}_a(1,2) + \frac{1}{2}\tilde{p}_a(1,1) + \tilde{p}_a(0,1) + \tilde{p}_a(0,2)\right] . \tag{31}$$

Obviously, it becomes unity only if the probabilities of spontaneous emission in b_2-mode (both in presence and in absence of photons in b_1-mode) are zero. Then all the photons present in the cavity would be in b_1-mode. However due to complex nature of time-dependence of $\tilde{p}_a(k,l)$'s, one does not find any periodicity in the variation of $F(t)$. The average number of "right" photons and of all photons in the evolved state of the two-atom cloner have been plotted as functions of time in the Fig. 5. A comparison of Figs. (5a) and (5b) shows tremendous improvement in cloning due to the cycling field.

V. AVERAGE FIDELITY OF CLONING FOR A FIXED BIASING FIELD

In the previous sections we had discussed the possibility of improvement in fidelity by changing the bias field as one changes the input state of the qubit. The question arises what is the fidelity of cloning for a fixed bias field. In such a case we have to work with the average of fidelity over all the input states of the qubit. To be precise let us consider the input state of the qubit as given by Eq. (5). We calculate the probability $\tilde{p}(k,l)$ as defined by (13). We next average this probability over all values of α and β: $\alpha = \cos(\chi/2), \beta = \sin(\chi/2)e^{i\varphi}$, where $0 \le \chi \le \pi$, $0 \le \varphi \le 2\pi$. Using average probabilities $\overline{p}(k,l)$ we obtain the average fidelity for all the states of the input qubit. The calculations are lengthy and we present results in the Fig. 6a. Note that for no bias, the fidelity is periodic in time whereas in presence of the bias field it is quasiperiodic. We conclude from this figure that there is considerable *improvement* in fidelity upto times of order $gt \sim 2$. This is also reflected by the behavior of the mean photon number in the "right" mode (Fig. 6b).

VI. CONCLUSIONS

In conclusion, we have proposed how the fidelity of a V-system-based quantum cloner can be improved by inhibiting the spontaneous emission effects on the unwanted transition. We showed that this can be done by applying a classical coherent field, which cycles the atom in the state $|e_2\rangle$ to some other metastable state. The fidelity remains close to unity over large intervals of time. Furthermore the fidelity of a cloning machine based on two atoms inside the cavity is much better provided we continue to use a cycling field. However we must add that in order to maintain the universality of the scheme we have to choose a cycling field which depends on the state of the input qubit. Otherwise for a fixed cycling field one has to be content with an average fidelity which for a range of interaction times is also found to be much better compared to the one in the absence of the cycling field.

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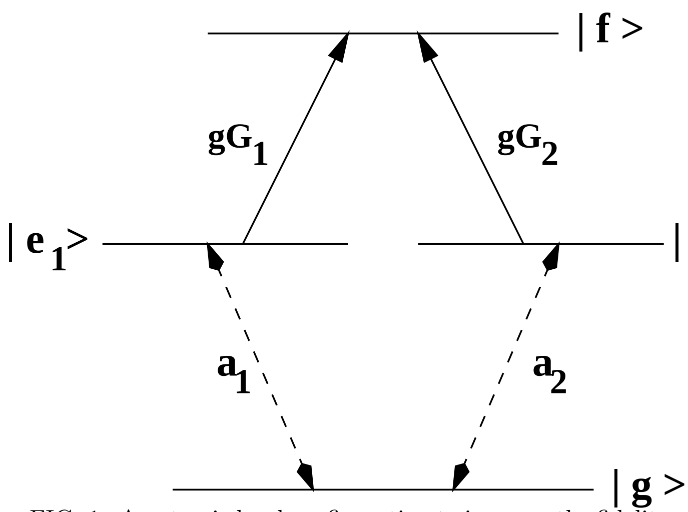

FIG. 1. An atomic level configuration to improve the fidelity of quantum cloning. Here the classical fields with Rabi frequency $2gG_i$ $(i = 1, 2)$ couple the levels $|e_1\rangle$ and $|e_2\rangle$ with the metastable state $|f\rangle$. The atom is inside a cavity, which allows only two field-modes a_1 and a_2.

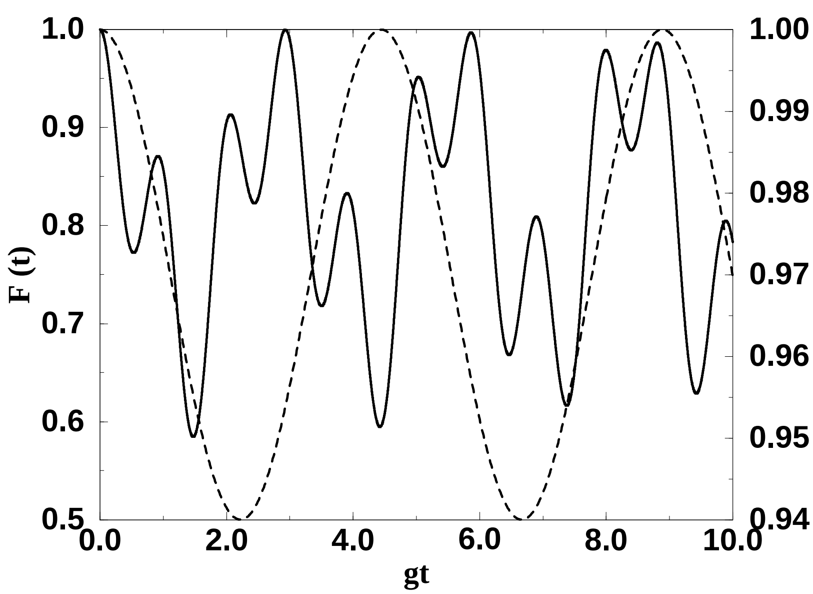

FIG. 2. These figures show the time-dependence of the fidelity of a four-level atomic cloner comprising a single atom for no cycling field ($G_1' = G_2' = 0$; dashed curve : tick levels are on left side) and in presence of external fields ($G_1' = 0$, $G_2' = 3$; solid curve : tick levels are on right side). It is obvious that the bias field improves the fidelity of cloning considerably.

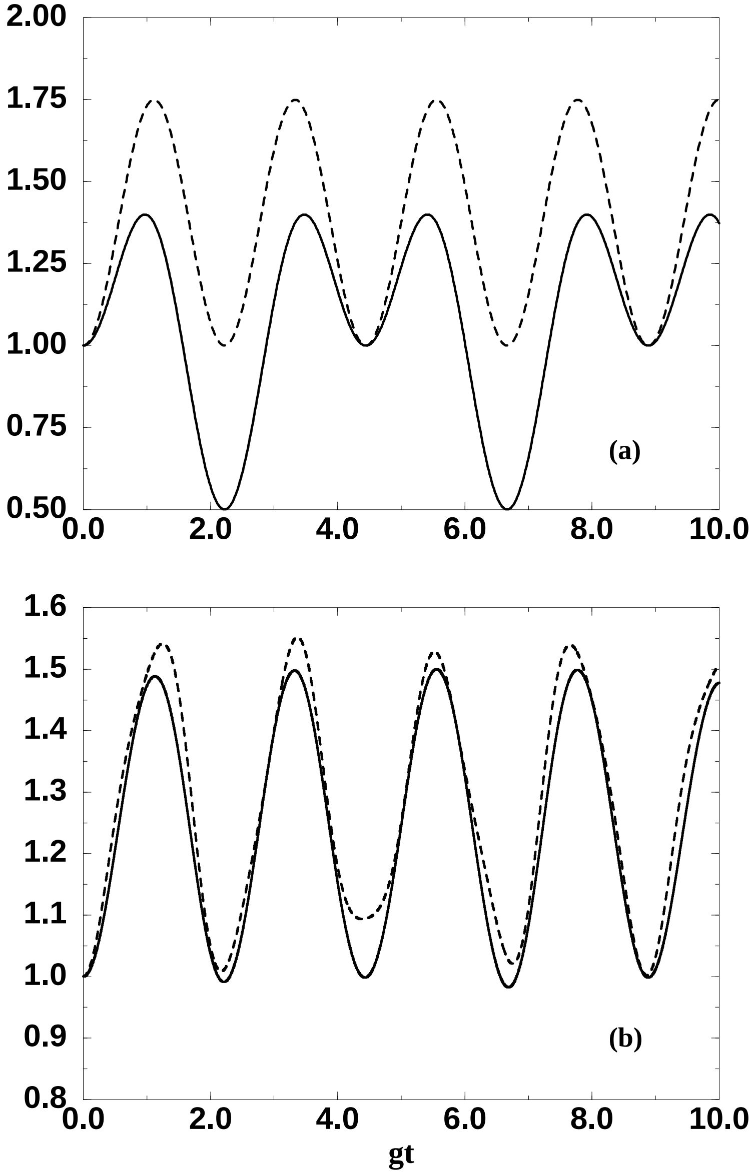

FIG. 3. These figures display the time-dependence of average number of photons in "right" mode, i.e., in b_1-mode (N_{right} ; solid curve) and of all photons (N_{all} ; dashed curve) in a single atom cloner under the conditions $G'_1 = G'_2 = 0$ [Fig. (a)] and $G'_1 = 0$, $G'_2 = 3$ [Fig. (b)]. It is seen that for non-zero G'_2, N_{right} and N_{all} approach each other. We have seen that for $G'_2 = 8$, i.e., for faster cycling of the population, they are nearly equal for almost all times.

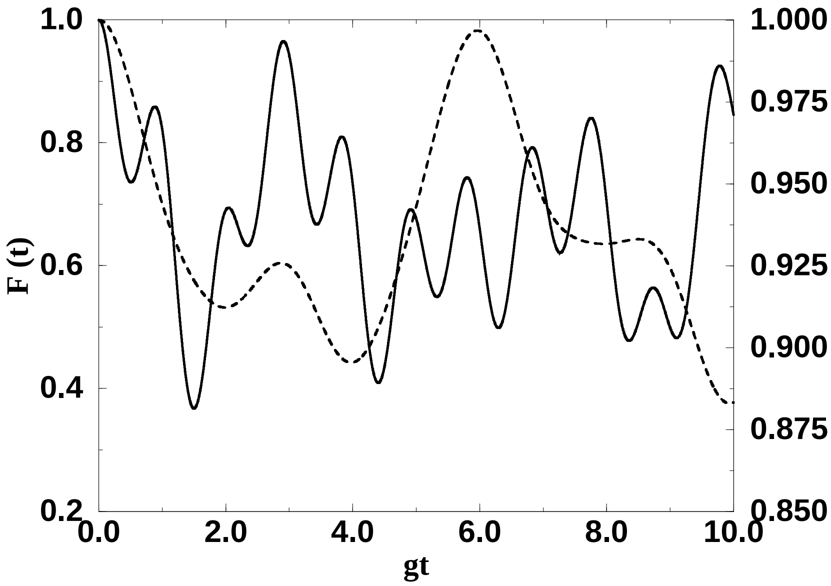

FIG. 4. The fidelity of a two-atom cloner is plotted as a function of time for no cycling field ($G'_1 = G'_2 = 0$; dashed curve : tick levels are on left side) and in presence of the bias field ($G'_1 = 0$, $G'_2 = 3$; solid curve : tick levels are on right side). Clearly the fidelity of cloning is improved by fast cycling of atomic population by a classical bias field.

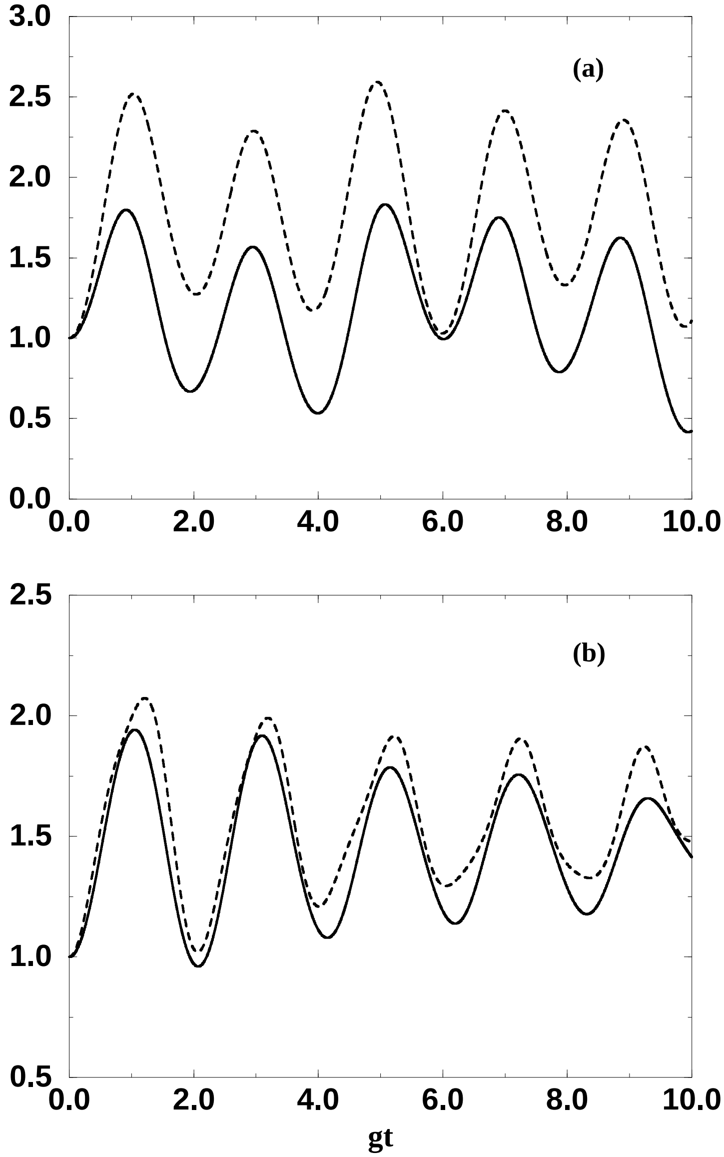

FIG. 5. The time-variation of average number of the photons in the b_1-mode ($N_{\rm right}$; solid curve) and all photons ($N_{\rm all}$; dashed curve) in a two-atom cloner have been displayed for $G_1' = G_2' = 0$ [Fig. (a)] and $G_1' = 0$, $G_2' = 3$ [Fig. (b)]. This shows that $N_{\rm right}$ and $N_{\rm all}$ become closer for larger value of G_2'.

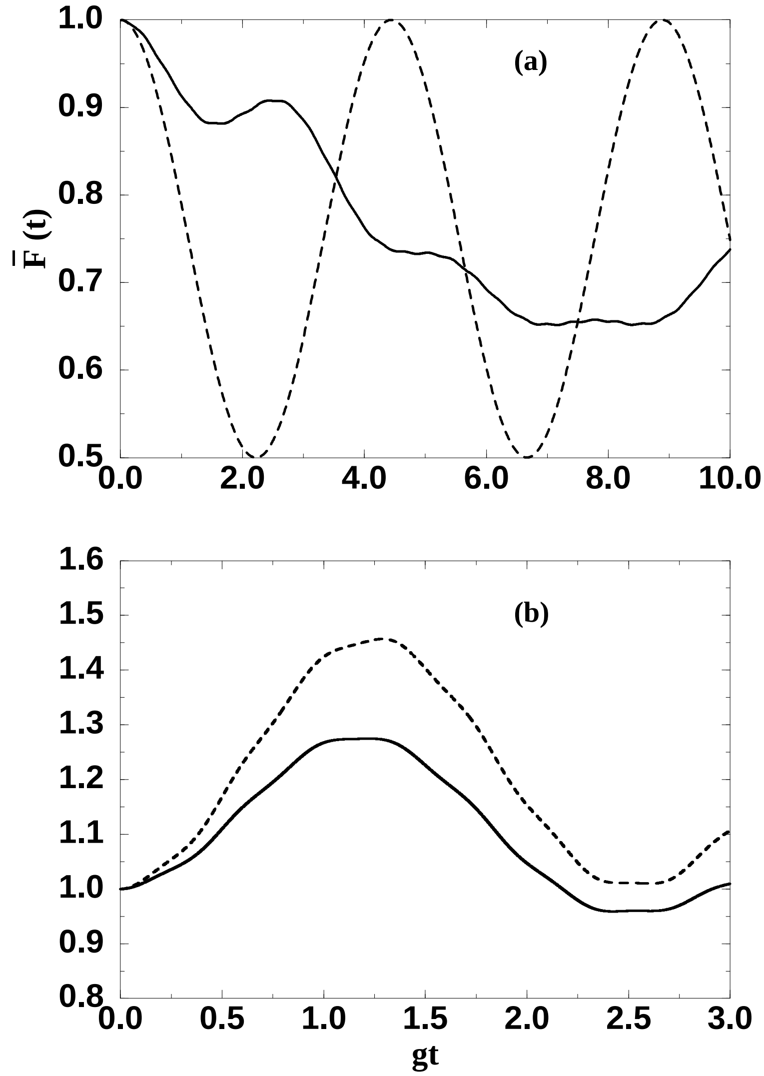

FIG. 6. (a) The fidelity of a single atom cloner averaged over all states of the input qubit is plotted as a function of time for the cases $G_1' = G_2' = 0$ (dashed curve) and $G_1' = 0\ ,\ G_2' = 8$ (solid curve). (b) The time-variation of number of "right" photons (solid curve) and all photons (dashed curve), averaged over all states of the input qubit are shown for some external field parameters $G_1' = 0\ ,\ G_2' = 8$.

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